8

Finite Element Methods

8.1 INTRODUCTION

The finite difference methods described in previous chapters can be considered as a direct discretization of differential equations. In finite element methods we generate difference equations by using approximate methods with the piecewise polynomial solution. The details of this formulation will be discussed, including a brief description of the weighted residual and the variational methods. We also discuss the construction of the piecewise polynomial functions in one, two and three space dimensions. Finally we study the application of the finite element methods to the solution of ordinary and partial differential equations.

8.2 WEIGHTED RESIDUAL METHODS

The weighted residual methods are the approximate methods which provide analytical procedure for obtaining solutions in the form of functions which are close in some sense to the exact solution of the boundary value problem or the initial value problem. We formulate the weighted residual methods for the boundary value problem

$$L[u] = r(\mathrm{x}), \ \mathrm{x} \in \mathcal{R} \tag{8.1}$$

$$U_{\mu}[u] = \gamma_{\mu}, \quad \mathrm{x} \in \partial\mathcal{R} \tag{8.2}$$

where $L[u]$ denotes a general differential operator involving spatial derivatives of u; $U_{\mu}[u]$ represents the appropriate number of boundary conditions and $\mathcal{R}$ is the domain with boundary $\partial\mathcal{R}$. The coordinate x is assumed as a one dimensional coordinate in the following section, although the definition of x may be extended and interpreted as a coordinate in multidimensional space. The solution of the boundary value problem (8.1)-(8.2) is often attempted by assuming an approximation to the solution $u(\mathrm{x})$, an expression of the form

$$u(\mathrm{x}) \approx w(\mathrm{x}, a_1, a_2, \cdots, a_N) \tag{8.3}$$

The size of one or more subdomains decreases as N is increased, with the result that the differential equation is satisfied on the average in smaller and smaller subdomains, and hopefully the residue approaches zero everywhere.

8.2.3 Galerkin method

In the Galerkin method the weighting function is chosen to be

$$W_j = \frac{\partial w(\mathbf{x}, \mathbf{a})}{\partial a_j}, \quad j = 1, 2, \ldots, N \tag{8.15}$$

where $w(\mathbf{x}, \mathbf{a})$ is the approximate solution of the problem. Equations (8.8) in the Galerkin method become

$$\int_{\mathcal{R}} \psi_j(\mathbf{x}) E(\mathbf{x}, \mathbf{a})\, d\mathbf{x} = 0, \quad j = 1, 2, \ldots, N \tag{81.6}$$

8.2.4 Moment method

In this method, we take the weighting function

$$W_j = P_j(\mathbf{x}) \tag{8.17}$$

where $P_j(\mathbf{x})$ are polynomials. Equations (8.8) become

$$\int_{\mathcal{R}} P_j(\mathbf{x}) E(\mathbf{x}, \mathbf{a})\, d\mathbf{x} = 0, \quad j = 1, 2, \ldots, N \tag{8.18}$$

The method of moments is similar to the Galerkin method except that the residual is made orthogonal to members of a system of functions which need not be the same as the approximating function. In practice, we take $W_j = \mathbf{x}^j$, and get better results if we orthogonalize them before use.

8.2.5 Collocation method

We choose N points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ in the domain $\mathcal{R}$ and define the weighting function as

$$W_j = \delta(\mathbf{x} - \mathbf{x}_j) \tag{8.19}$$

where δ represents the unit *impulse* or *Dirac delta* which vanishes everywhere except at $\mathbf{x} = \mathbf{x}_j$. The collocation equations become

$$\int_{\mathcal{R}} \delta(\mathbf{x} - \mathbf{x}_j) E[\mathbf{x}, \mathbf{a}]\, d\mathbf{x} = 0 \tag{8.20}$$

which can be written as

$$E[\mathbf{x}_j, \mathbf{a}] = 0, \quad j = 1, 2, \ldots, N \tag{8.21}$$

This criterion is thus equivalent to putting $E[\mathbf{x}, \mathbf{a}]$ equal to zero at N points in the domain $\mathcal{R}$. The distribution of the collocation points on $\mathcal{R}$ is arbitrary. However, in practice we distribute the collocation points uniformly on $\mathcal{R}$.

Example 8.1 Consider the boundary value problem

$$u'' + (1+x^2)u + 1 = 0$$
$$u(\pm 1) = 0$$

Determine the coefficients of the approximate solution

$$w(x) = a_1(1-x^2) + a_2x^2(1-x^2)$$

by using various weighted residual methods.

Substituting the approximate solution in the differential equation, we get

$$E[\mathbf{x}, \mathbf{a}] = 1 + a_1(-1-x^4) + a_2(2-11x^2-x^6)$$

Since the approximate solution satisfies the condition of symmetry, we consider the boundary value problem in the interval [0, 1].

Least square method

Equation (8.11) becomes

$$WE = \frac{68}{45}a_1^2 + \frac{7096}{1155}a_1a_2 + \frac{63404}{4095}a_2^2 - \frac{12}{5}a_1 - \frac{76}{21}a_2 + 1$$

which leads to the linear equations

$$\frac{68}{45}a_1 + \frac{3548}{1155}a_2 = \frac{6}{5}$$
$$\frac{3548}{1155}a_1 + \frac{63404}{4095}a_2 = \frac{38}{21}$$

On solving them, we find

$$a_1 = 0.932718, \quad a_2 = -0.068181$$

Partition method

We divide the interval [0, 1] into two subintervals [0, 1/2] and [1/2, 1]. Equations (8.14) give

$$\frac{81}{160}a_1 - \frac{1453}{2688}a_2 = \frac{1}{2},$$
$$\frac{111}{160}a_1 + \frac{6317}{2688}a_2 = \frac{1}{2}$$

Solving these equations, we find

$$a_1 = 0.923680, \qquad a_2 = -0.059914$$

Galerkin method

Equations (8.16) become

$$\int_0^1 (1-x^2)[1 + a_1(-1-x^4) + a_2(2-11x^2-x^6)]\ dx = 0$$
$$\int_0^1 (x^2-x^4)[1 + a_1(-1-x^4) + a_2(2-11x^2-x^6)]\ dx = 0$$

which on simplification may be written as

$$\begin{bmatrix} \frac{10120196}{45045} & -\frac{890871744}{2837835} \\ -\frac{890871744}{2837835} & \frac{16231424}{36855} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} -\frac{634}{105} \\ \frac{608}{63} \end{bmatrix}.$$

We obtain

$$u_0 = 0.932718, \quad u_1 = 0.686755$$

Example 8.3 Consider the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(x, 0) = \cos\frac{\pi x}{2} \qquad -1 \leqslant x \leqslant 1$$

$$u(-1, t) = u(1, t) = 0$$

Use the Galerkin method with the approximate solution of the form

$$w(x, t) = (1 - x^2)(1 - 4x^2)u_0(t) + \frac{16}{3}(x^2 - x^4)u_1(t)$$

where $u_0(t)$ and $u_1(t)$ are the unknown solution values at the nodes 0 and 1/2 respectively, to reduce the partial differential equation to a simultaneous set of two ordinary differential equation.

The Galerkin equations (8.16) become

$$\int_0^1 \mathbf{N}^T[\mathbf{N}''(x)\boldsymbol{\phi}(t) - \mathbf{N}(x)\dot{\boldsymbol{\phi}}]\, dx = 0$$

or

$$\int_0^1 \left\{ \begin{bmatrix} N_0 N_0'' & N_0 N_1'' \\ N_1 N_0'' & N_1 N_1'' \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} - \begin{bmatrix} N_0 N_0 & N_0 N_1 \\ N_1 N_0 & N_1 N_1 \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \end{bmatrix} \right\} dx = 0$$

where

$$\mathbf{N} = [N_0 \quad N_1], \; \boldsymbol{\phi} = [u_0 \quad u_1]^T$$

$$N_0(x) = (1 - 5x^2 + 4x^4), \; N_1(x) = \frac{16}{3}(x^2 - x^4)$$

$$\dot{\boldsymbol{\phi}} = \left[\frac{du_0}{dt} \; \frac{du_1}{dt}\right]^T \text{ and } \mathbf{N}'' = \left[\frac{d^2 N_0}{dx^2} \; \frac{d^2 N_1}{dx^2}\right]$$

Simplifying we get the following ordinary differential equations

$$\begin{bmatrix} \frac{104}{315} & -\frac{128}{945} \\ -\frac{128}{945} & \frac{2048}{2835} \end{bmatrix} \begin{bmatrix} \dot{u}_0 \\ \dot{u}_1 \end{bmatrix} = \begin{bmatrix} -\frac{620}{105} & \frac{2368}{315} \\ \frac{2368}{315} & -\frac{11264}{945} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

which may be written as

$$\dot{\boldsymbol{\phi}} = \mathbf{A}\boldsymbol{\phi}$$

where

$$\mathbf{A} = \frac{1}{12}\begin{bmatrix} -177 & 208 \\ \frac{1467}{16} & -159 \end{bmatrix}$$

The initial conditions are given by

$$u_0(0) = 1, \quad u_1(0) = \frac{1}{\sqrt{2}}$$

Example 8.4 Consider the boundary value problem

$$\nabla^2 u = -1, \quad |x| \leqslant 1, \quad |y| \leqslant 1$$
$$u = 0, \quad |x| = 1, \quad |y| = 1$$

Use the Galerkin method to determine the solution values at the nodes (0, 0), ($\frac{1}{2}$, 0) and ($\frac{1}{2}$, $\frac{1}{2}$).

It may be noted that the solution of the boundary value problem satisfies the symmetry conditions

$$u(-x, y) = u(x, y), \; u(x, -y) = u(x, y)$$
$$u(y, x) = u(x, y)$$

Thus, there are three mesh points 0, 1 and 2 as shown in Figure 8.1

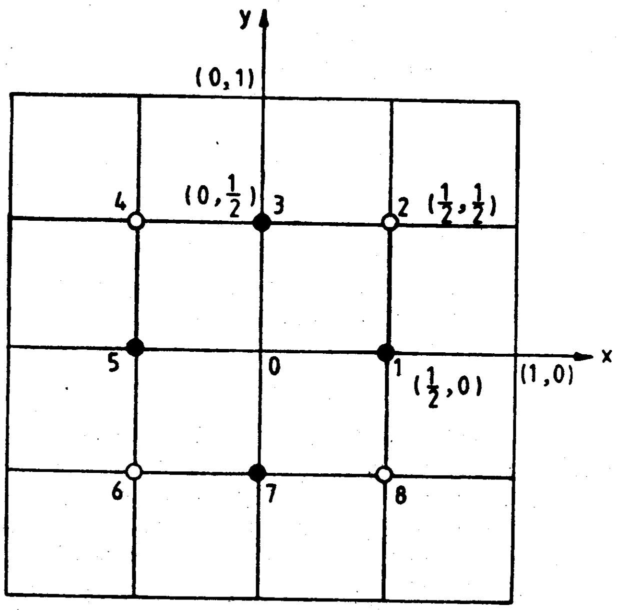

Fig. 8.1 Mesh points

also satisfy the equations (8.35 ii) and (8.35 iii) and are called *natural* or *suppressible* boundary conditions.

Finally, we consider the integral

$$J = \iint\limits_{\mathcal{R}} F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) dx\, dy \tag{8.36}$$

over the area $\mathcal{R}$ enclosed by the curve $\partial\mathcal{R}$. The equation (8.31) becomes

$$\delta J = \iint\limits_{\mathcal{R}} \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y\right) dx\, dy \tag{8.37}$$

where $\quad u_x = \dfrac{\partial u}{\partial x}$ and $u_y = \dfrac{\partial u}{\partial y}$

To simplify (8.37) further we make use of the following theorem.

THEOREM (Green) 8.1 Let $\mathcal{R}$ be a closed finite region of the (x, y) plane bounded by a piecewise smooth curve $\partial\mathcal{R}$ without double point. If the functions P and Q are continuous with continuous first partial derivatives in $\mathcal{R}$ then

$$\int\limits_{\partial\mathcal{R}} (P\, dx + Q\, dy) = \iint\limits_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy \tag{8.38}$$

Substituting $P = -\delta u \dfrac{\partial F}{\partial u_y}$ and $Q = \delta u \dfrac{\partial F}{\partial u_x}$ in (8.38) we obtain

$$\begin{aligned}
&\int\limits_{\partial\mathcal{R}} \left(-\frac{\partial F}{\partial u_y}\delta u\, dx + \frac{\partial F}{\partial u_x}\delta u\, dy\right) \\
&= \iint\limits_{\mathcal{R}} \left[\frac{\partial}{\partial x}\left(\delta u \frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\delta u \frac{\partial F}{\partial u_y}\right)\right] dx\, dy \\
&= \iint\limits_{\mathcal{R}} \left(\delta u_x \frac{\partial F}{\partial u_x} + \delta u_y \frac{\partial F}{\partial u_y}\right) dx\, dy \\
&\quad + \iint\limits_{\mathcal{R}} \left(\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right)\delta u\, dx\, dy
\end{aligned}$$

The equation (8.37) becomes

$$\begin{aligned}
\delta J = &\iint\limits_{\mathcal{R}} \left[\frac{\partial F}{\partial u} - \frac{\delta}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)\right] \delta u\, dx\, dy \\
&+ \int\limits_{\partial\mathcal{R}} \left(-\frac{\partial F}{\partial u_y} dx + \frac{\partial F}{\partial u_x} dy\right)\delta u = 0
\end{aligned} \tag{8.39}$$

This is satisfied if the following conditions hold,

$$\text{(i)} \quad \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

$$\text{(ii)} \quad \int_{\partial \mathcal{R}} \left(-\frac{\partial F}{\partial u_y} dx + \frac{\partial F}{\partial u_x} dy\right) \delta u = 0 \tag{8.40}$$

The equation (8.40 i) is the *Euler* equation. If u is prescribed on $\partial\mathcal{R}$ i.e. $\delta u = 0$ then the equation (8.40 ii) is satisfied otherwise when u is not specified on $\partial\mathcal{R}$, we have

$$-\frac{\partial F}{\partial u_y} \cos \nu + \frac{\partial F}{\partial u_x} \sin \nu = 0 \tag{8.41}$$

where ν is the angle which the outward normal to the boundary $\partial\mathcal{R}$ makes with the x axis.

The conditions (8.41) are called *natural* or *suppressible* boundary conditions.

8.3.1 Ritz method

In order to solve a given boundary value problem by the *Ritz* method, we try to write the differential equation as the Euler equation of some variational problem. This will give the appropriate expression for $J[u]$. We now reduce this variational problem to a simple minimizing problem by assuming an approximate function in the form (8.5). Substituting (8.5) in (8.22), we get $J[w]$ as a function of the unknowns $a_1, a_2, \ldots, a_N$. For minimizing $J[w]$, we have

$$\frac{\partial J[w]}{\partial a_j} = \int_a^b \left(\frac{\partial F}{\partial w'}\psi_j' + \frac{\partial F}{\partial w}\psi_j\right) dx = 0, \; j = 1, 2, \ldots, N \tag{8.42}$$

which gives N equations in N unknowns. If $\psi_j(x)$ possess continuous second order derivatives, then integrating by parts the first term in the integrand of (8.42) we get

$$\int_a^b \psi_j\left[-\frac{d}{dx}\left(\frac{\partial F}{\partial w'}\right) + \frac{\partial F}{\partial w}\right] dx = 0, \; j = 1, 2, \ldots, N \tag{8.43}$$

Equations (8.43) are identical with the Galerkin equations (8.16) for differential equations, which are identical with the Euler equation (8.28). For the differential equation

$$-\frac{d}{dx}(pu') + qu = r(x) \tag{8.44}$$

with boundary conditions (8.23), it can be easily verified that with

$$F = pu'^2 + qu^2 - 2ru$$

the end points x_i and x_{i+1} such that the length of the element (e) is unit may be written as

$$x = x_i + (x_{i+1} - x_i)\xi$$
$$= (1-\xi)x_i + \xi x_{i+1} \tag{8.49}$$

From (8.49) and (8.47), we get

(i) $\xi = \dfrac{x - x_i}{x_{i+1} - x_i} = N_{i+1}(x)$

and

(ii) $1 - \xi = \dfrac{x_{i+1} - x}{x_{i+1} - x_i} = N_i(x)$ (8.50)

The transformation (8.49) transforms or maps an element (e) along the x-axis into a *standard interval* [0, 1]. Similarly, if we choose the mid-point of the element (e) as the origin of the ξ-axis then the transformation

$$x = \frac{1}{2}(x_i + x_{i+1}) + \frac{1}{2}l^{(e)}\xi \tag{8.51}$$

maps the subinterval $[x_i, x_{i+1}]$ into a standard interval $[-1, 1]$, where $l^{(e)} = x_{i+1} - x_i$ is the length of the element (e).

The functions ξ and $(1-\xi)$ in (8.50) are ratios of lengths and are called *length, local* or *natural coordinates*. We denote $(1-\xi)$ and ξ by L_i and L_{i+1}, respectively. The coordinates $L_i(x)$ and $L_{i+1}(x)$ are not independent since we have

$$L_i(x) + L_{i+1}(x) = 1 \tag{8.52}$$

The equation (8.49) can also be written as

$$x = L_i(x)x_i + L_{i+1}(x)x_{i+1} \tag{8.53}$$

which shows that the mapping (8.49) is also an interpolation scheme that gives the x coordinate of any point on the element (e) when the corresponding L_i and L_{i+1} coordinates are known. The variation of (L_i, L_{i+1}) inside the element (e) is shown in Figure 8.2(b). Using (8.52) and (8.53), we obtain

$$\begin{bmatrix} L_i \\ L_{i+1} \end{bmatrix} = \begin{bmatrix} x_i & x_{i+1} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} = \frac{1}{(x_{i+1} - x_i)} \begin{bmatrix} -1 & x_{i+1} \\ 1 & -x_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} N_i \\ N_{i+1} \end{bmatrix}$$

Thus we find that in the case of the linear piecewise approximate function the local coordinates are also the shape functions.